Profinite Rigidity

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The Basic Question

Question

To what extent can we determine a group Γ from its set of finite quotients $\mathcal{C}(\Gamma)$?

No Finite Quotients

$$G_4 = \langle \alpha, \beta, \gamma, \delta \mid \beta\alpha = \alpha^2\beta, \gamma\beta = \beta^2\gamma, \delta\gamma = \gamma^2\delta, \alpha\delta = \delta^2\alpha \rangle$$

$$B_3 = \langle a, b, c, d \mid ba^2 = a^3b, dc^2 = c^3d, [a, b] = d, [c, d] = b \rangle$$

Residually Finite Groups

Definition

We say a group Γ is *residually finite*, if for each $\gamma \in \Gamma \setminus \{1\}$, there exists a group homomorphism $\pi : \Gamma \to$ Finite, such that $\pi(\gamma) \neq 1$.

- The fundamental group of compact 3-manifolds.
- Free groups.
- Finite groups.
- Finitely generated linear groups.

Rigidity

$$\begin{array}{ccc} X & \overset{\text{then } \cong}{\dashrightarrow} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\text{if } \simeq} & Y' \end{array}$$

Mostow Rigidity

Theorem (Mostow's Rigidity Theorem)

Suppose M and N are complete finite-volume hyperbolic manifolds of dimension $n \geq 3$. If there exists an isomorphism $f : \pi_1(M) \to \pi_1(N)$, then it is induced by a unique isometry from M to N.

Profinite Groups: Axiomatic

Definition

We say a topological group G is a *profinite group*, if G is compact, totally disconnected, and Hausdorff.

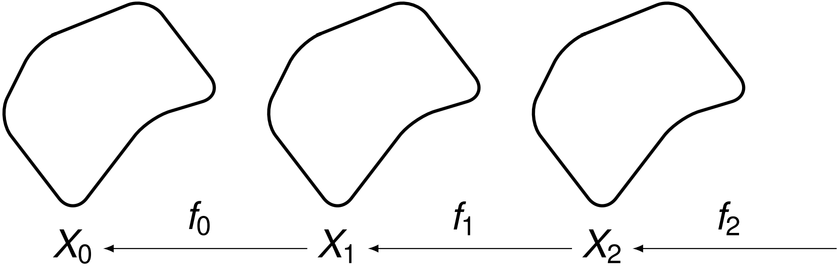
Example

Let G be a finite group endowed with the discrete topology. Then, G is a profinite group.

Inverse Systems

Definition

Let $(I, \leq)$ be a directed set. An *inverse system*, is a family of spaces X_i and a family of maps $f_{ij} : X_i \to X_j$ whenever $i \geq j$, such that $f_{ii} = \mathrm{id}_{X_i}$ and $f_{jk} \circ f_{ij} = f_{ik}$ whenever $i \geq j \geq k$. Denote by $\langle X_i, f_{ij}, I \rangle$, our inverse system.

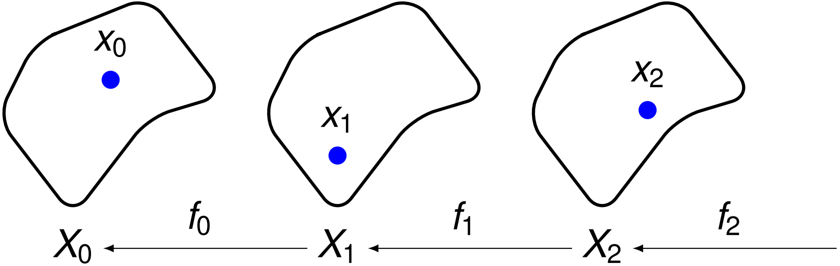


Inverse Limits

Definition

The *inverse limit* of our inverse system $\langle X_i, f_{ij}, I\rangle$, is the set

$$\varprojlim X_i := \left\{ (x_i) \in \prod_{i\in I} X_i \;\middle|\; f_{ij}(x_i) = x_j, \forall i \geq j \right\}.$$



Profinite Groups: Constructive

Definition

We say a topological group G is a *profinite group*, if G is isomorphic to the inverse limit of an inverse system of discrete finite groups.

Example

The group of p-adic integers $\mathbb{Z}_p$ under addition is a profinite group, where $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Profinite Completion

Definition

The *profinite completion* of a group Γ, denoted $\widehat{\Gamma}$, is the inverse limit of $\langle \Gamma/N_i, \phi_{ij}, I \rangle$, where N_i are normal subgroups of Γ of finite index, $i \geq j$ if, and only if, $N_i \leq N_j$, and for $i \geq j$, $\phi_{ij} : \Gamma/N_i \to \Gamma/N_j$ is defined by $\phi_{ij}(gN_i) = gN_j$. That is to say,

$$\widehat{\Gamma} := \varprojlim \Gamma/N_i \qquad [\Gamma : N_i] < \infty.$$

Profinite Completion

Lemma

The inclusion map $\iota : \Gamma \to \widehat{\Gamma}$ defined by $g \mapsto (gN_i)$ is one-to-one if, and only if, Γ is residually finite.

It turns out that $\iota(\Gamma)$ is dense in $\widehat{\Gamma}$. Other properties include, for $\Delta \leq \Gamma$:

- $\Delta \lhd \Gamma$ if, and only if, $\overline{\iota(\Delta)} \lhd \widehat{\Gamma}$.
- $[\Gamma : \Delta] = [\widehat{\Gamma} : \overline{\iota(\Delta)}]$.
- If $\Delta \lhd \Gamma$, then $\Gamma/\Delta \cong \widehat{\Gamma}/\overline{\iota(\Delta)}$.

Transforming The Problem

Theorem (Nikolov and Segal)

If Γ is finitely generated, then every group homomorphism from $\widehat{\Gamma}$ to a profinite group is continuous.

With the following observation, we can translate our question on finite quotients to that of profinite completions.

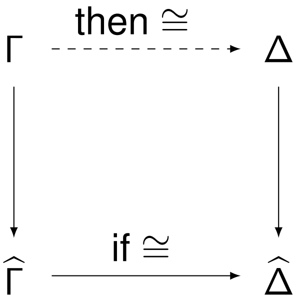
Theorem

For finitely generated groups Γ_1 and Γ_2, $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ if, and only if, $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$.

Profinite Rigidity

Definition

We say a finitely generated, residually finite group Γ is *profinitely rigid in the absolute sense*, if $\widehat{\Gamma} \cong \widehat{\Delta}$ implies $\Gamma \cong \Delta$.



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- Baumslag (1974): There exists $\Gamma = (\mathbb{Z}/25) \rtimes_\alpha \mathbb{Z}$ and $\Delta = (\mathbb{Z}/25) \rtimes_\beta \mathbb{Z}$ such that $\widehat{\Gamma} \cong \widehat{\Delta}$, but $\Gamma \ncong \Delta$.

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- $N \times \mathbb{Z}$, where N is finite.
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- $F_2 \times F_2$ is not profinitely rigid.

The Big Open Question

Question (Remeslennikov)

Are the free groups F_n of rank $n \geq 2$ profinitely rigid?

Positive Result: Relative Rigidity

Theorem (Bridson-Conder-Reid, 2016)

Let Γ be a finitely generated Fuchsian group, i.e., a lattice in $\mathrm{PSL}(2, \mathbb{R})$, and let Λ be a lattice in a connected Lie group. If $\widehat{\Gamma} \cong \widehat{\Lambda}$, then $\Gamma \cong \Lambda$.

Arithmetic Kleinian Groups

- The Bianchi group $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ with $\omega^2 + \omega + 1 = 0$ is profinitely rigid in the absolute sense.
- Non-uniform lattice of minimal co-volume in $\mathrm{PSL}(2, \mathbb{C})$ is profinitely rigid in the absolute sense.
- The fundamental group of the Weeks manifold (the closed hyperbolic 3-manifold of minimal volume) is profinitely rigid in the absolute sense.

Thank you all for your attention!